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A conjecture on the zeros of Bessel functions

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We denote by $j_{\nu\kappa}$, $\kappa = k - \alpha/\pi$, $k = 1, 2, \dots$, the zeros of the cylinder function

$$C_\nu(x) = J_\nu(x) \cos \alpha - Y_\nu(x) \sin \alpha, \quad 0 \leq \alpha < \pi,$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and the second kind respectively. See [1] for more details on this notation. We observe that when $\alpha = 0$, we get $j_{\nu\kappa} = j_{\nu k}$, i.e., the zeros of $J_\nu(x)$.

With this notation for the zeros of $C_\nu(x)$, the well-known McMahon asymptotic expansion, can be written in the following form:

$$j_{\nu\kappa} = \beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}} - \frac{4(\mu - 1)(7\mu - 31)}{3(8\beta_{\nu\kappa})^3} + \dots,$$

where $\beta_{\nu\kappa} = (\kappa + \nu/2 - \frac{1}{2})\pi$, $\mu = 4\nu^2$. We know [3, p. 506] that

$$j_{\nu\kappa} \geq \beta_{\nu\kappa}, \quad |\nu| \leq \frac{1}{2}.$$

Moreover, Förster and Petras [2] proved, among other things, that

$$\beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}} - \frac{4(\mu - 1)(7\mu - 31)}{3(8\beta_{\nu\kappa})^3} \leq j_{\nu\kappa} \leq \beta_{\nu\kappa} - \frac{\mu - 1}{8\beta_{\nu\kappa}}, \quad |\nu| \leq -\frac{1}{2}.$$

We conjecture that for $|\nu| \leq \frac{1}{2}$, an even (odd) number of terms of McMahon's expansion always gives upper (lower) bounds for $j_{\nu\kappa}$.

References

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- [3] G.N. Watson, A Treatise on the Theory of Bessel Functions, 2nd Edition, Cambridge University Press, 1944.

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